

A SETTING FOR HIGHER ORDER DIFFERENTIAL EQUATIONS FIELDS AND HIGHER ORDER LAGRANGE AND FINSLER SPACES

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ABSTRACT. We use the Frölicher-Nijenhuis formalism to reformulate the inverse problem of the calculus of variations for a system of differential equations of order $2k$ in terms of a semi-basic 1-form of order k . Within this general context, we use the homogeneity proposed by Crampin and Saunders in [14] to formulate and discuss the projective metrizability problem for higher order differential equation fields. We provide necessary and sufficient conditions for higher order projective metrizability in terms of homogeneous semi-basic 1-forms. Such a semi-basic 1-form is the Poincaré-Cartan 1-form of a higher order Finsler function, while the potential of such semi-basic 1-form is a higher order Finsler function.

1. INTRODUCTION

The Frölicher-Nijenhuis formalism is a very useful tool for developing a differential calculus that provides a geometric setting for studying differential equations fields, [4, 5, 12, 15, 19, 29, 31].

The framework for studying higher order differential equations fields, on a configuration manifold M , is the higher order tangent bundle $T^r M$, for some $r \geq 1$. In Section 2 we discuss some geometric structures that naturally live on higher order tangent bundles: vertical distributions, Liouville vector fields, tangent structures. We use the Frölicher-Nijenhuis formalism and these geometric structures to provide a vertical differential calculus, which is very useful for studying higher order differential equations fields. Motivated by the foliated structure of the higher order tangent bundles, we show that vertical vector fields, as well as their dual, semi-basic 1-forms, play an important role in the vertical differential calculus we associate to a higher order differential equation field. We will use the formalism developed in Subsection 2.2 and especially Lemma 2.3 in Sections 3 and 4 to characterize those differential equations fields that may be associated to a variational problem of a Lagrange or a Finsler space of higher order.

The inverse problem of the calculus of variations requires to determine the necessary and sufficient conditions such that a system of ordinary differential equations, of order $2k$, may be derived from a variational problem. For $k = 1$, these conditions can be formulated in terms of a multiplier matrix [17, 25, 26, 27, 28], a closed 2-form [2, 10], or a semi-basic 1-form [5]. The approach, based on the existence of a closed 2-form, developed by Crampin in [10], was extended by de León and Rodrigues in [20] for $k > 1$. In Section 3 we use the vertical differential calculus, which we develop in Section 2, to provide global formulations for the geometric structures one can associate to higher order Lagrangians and higher order differential equations fields. In Theorem 3.4 we reformulate the inverse problem of the calculus of variations in terms of a semi-basic 1-form of order k . For the variational case, we show that such a semi-basic 1-form is the Poincaré-Cartan 1-form of a Lagrangian of order k . In Proposition 3.6 we prove that some homogeneity properties of a regular Lagrangian transfer to its canonical Euler-Lagrange vector field.

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An important case of the inverse problem of the calculus of variations refers to homogeneous systems of ordinary differential equations. For $k = 1$, this problem contains what is known as the projective metrizable problem or "the Finslerian version of Hilbert's fourth problem", [1, 11, 32]. The projective metrizable problem requires to determine if the solutions of a homogeneous system of second order ordinary differential equations coincide with the geodesics of a Finsler metric, up to an orientation preserving reparameterization [7, 8, 30, 33]. For the case $k > 1$, an attempt to address and study the projective metrizable problem, requires first a good definition of homogeneity for systems of higher order differential equations as well as for higher order Lagrangians. In this work we use the definitions of homogeneity proposed by Crampin and Saunders in [14] to formulate and study the projective metrizable problem in Section 4. In Subsection 4.1 we introduce and discuss higher order Finsler spaces. We show that the regularity condition, which we propose for a higher order Finsler function, is equivalent to the regularity condition proposed by Crampin and Saunders in [14] for parametric Lagrangians and that it reduces, when $k = 1$, to the classic regularity condition of a Finsler function. We show that the variational problem of a higher order Finsler function uniquely determines a projective class of homogeneous differential equations fields. In Theorem 4.5 we characterize the projective metrizable problem of a homogeneous differential equation field of order $2k$ in terms of a homogeneous semi-basic 1-form of order k . We prove that, similarly with what happens in the case $k = 1$, such a semi-basic 1-form is the Poincaré-Cartan 1-form of a Finsler function of order k . Moreover, the potential of such homogeneous semi-basic 1-form is a Finsler function of order k that metricizes the equation field.

In the last section we discuss some examples of higher order differential equations fields and their relations with higher order Lagrange and Finsler spaces. It has been shown in [9] that biharmonic curves, which are solutions of a fourth order differential equations field, are solutions of the Euler-Lagrange equations for a regular Lagrangian L_2 of order 2. See also [4] for a different approach. We use the homogeneity properties of the second order Lagrangian L_2 to obtain some information for the corresponding Euler-Lagrange vector field (biharmonic differential equations field). We provide an example of a second order Finsler functions, which in the Euclidian context reduces to the parametric Lagrangian L , studied by Crampin and Saunders in [14, §6].

2. VERTICAL DIFFERENTIAL CALCULUS ON HIGHER ORDER TANGENT BUNDLES

In this section we discuss first some geometric structures that are naturally defined on higher order tangent spaces: vertical distributions, Liouville vector fields, tangent structures, higher order differential equations fields. We use these geometric structures and the corresponding differential calculus induced by the Frölicher-Nijenhuis formalism to develop a geometric setting, which we will use in Sections 3 and 4 to discuss two important problems associated to a (homogeneous) higher order differential equation field.

2.1. Geometric structures on higher order tangent bundles. In this work M is a real, n -dimensional and C^∞ -smooth manifold. We will assume that all objects are smooth where defined. We denote the ring of smooth functions on M by $C^\infty(M)$, while the Lie algebra of vector fields on M is denoted by $\mathfrak{X}(M)$.

The framework to develop a geometric setting for studying systems of higher order ordinary differential equations on a manifold M is the higher order tangent bundle $T^r M = J_0^r M$, for some $r \geq 1$, [2, 12, 19, 24, 34]. This is the jet bundle of order r , of curves c from a neighborhood of 0 in \mathbb{R} to M . For a curve $c : I \rightarrow M$, $c(t) = (x^i(t))$, consider $j^r c : I \rightarrow T^r M$ its jet lift of order r . If (x^i) are local coordinates on M , the induced local coordinates on $T^r M$ are denoted by

$(x^i, y^{(1)i}, \dots, y^{(r)i})$, where

$$y^{(\alpha)i}(j_0^r c) = \frac{1}{\alpha!} \frac{d^\alpha (x^i(c(t)))}{dt^\alpha} \Big|_{t=0}, \quad \alpha \in \{1, \dots, r\}.$$

Let $y^{(0)i} := x^i$ and denote $T^0 M = M$. The canonical submersion $\pi_\alpha^r : T^r M \rightarrow T^\alpha M$, for each $\alpha \in \{0, 1, \dots, r-1\}$, induces a natural foliation of $T^r M$. We will consider also the subbundle $T_0^r M = \{(x, y^{(1)}, \dots, y^{(r)}) \in T^r M, y^{(1)} \neq 0\}$. It follows that $T_0^r M \setminus \{0\} = (\pi_1^r)^{-1}(T_0 M)$.

A curve $c : I \rightarrow M$ is called a *regular curve* if $j^r c(t) \in T_0^r M$ for all $t \in I$.

The *tangent structure* (or vertical endomorphism) of order r is the $(1, 1)$ -type tensor field on $T^r M$ defined as

$$(2.1) \quad J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \dots + \frac{\partial}{\partial y^{(r)i}} \otimes dy^{(r-1)i}.$$

It has the following properties: $J^{r+1} = 0$, $\text{Im } J^\alpha = \text{Ker } J^{r-\alpha+1}$, $\alpha \in \{1, \dots, r\}$.

The foliated structure of $T^r M$ gives rise to r regular vertical distributions

$$V_\alpha(u) = \text{Ker } D_u \pi_{\alpha-1}^r = \text{Im } J_u^\alpha = \text{Ker } J_u^{r-\alpha+1}, \quad \text{for } u \in T^r M, \alpha \in \{1, \dots, r\}.$$

Each distribution V_α , for $\alpha \in \{1, \dots, r\}$, is tangent to the fibers of $\pi_{\alpha-1}^r : (x^i, y^{(1)i}, \dots, y^{(k)i}) \rightarrow (x^i, y^{(1)i}, \dots, y^{(\alpha-1)i})$, and hence it is integrable. We have that $\dim V_\alpha = (r - \alpha + 1)n$, $\alpha \in \{1, \dots, r\}$ and $V_r(u) \subset V_{r-1}(u) \subset \dots \subset V_1(u)$, for each $u \in T^r M$. We will denote by $\mathfrak{X}^{V_\alpha}(T^r M)$ the Lie subalgebra of vertically valued vector fields.

An important set of vertical vector fields is provided by the *Liouville vector fields* (or dilatation vector fields) $\mathbb{C}_\alpha \in \mathfrak{X}^{V_\alpha}(T^r M)$, $\alpha \in \{1, \dots, r\}$. These vector fields are locally given by:

$$(2.2) \quad \mathbb{C}_\alpha = y^{(1)i} \frac{\partial}{\partial y^{(\alpha)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(\alpha+1)i}} + \dots + (r+1-\alpha)y^{(r+1-\alpha)i} \frac{\partial}{\partial y^{(r)i}}, \quad \alpha \in \{1, \dots, r\}.$$

For the Liouville vector fields, we have the following formulae for their Lie brackets

$$(2.3) \quad [\mathbb{C}_\alpha, \mathbb{C}_\beta] = \begin{cases} (\alpha - \beta)\mathbb{C}_{\alpha+\beta-1}, & \text{if } \alpha + \beta \leq r-1, \\ 0, & \text{otherwise.} \end{cases}$$

For a vector valued l -form L on $T^r M$ consider the *inner product* $i_L : \Lambda^q(T^r M) \rightarrow \Lambda^{q+l-1}(T^r M)$ and the *exterior derivative* $d_L : \Lambda^q(T^r M) \rightarrow \Lambda^{q+l}(T^r M)$, see [16, II.8]. These two derivations, of order $(l-1)$ and respectively l , are related by the following formula

$$(2.4) \quad d_L = i_L \circ d + (-1)^l d \circ i_L.$$

For two vector valued forms K and L on $T^r M$, of degree k and respectively l , consider the *Frölicher-Nijenhuis bracket* $[K, L]$, which is the vector valued $(k+l)$ -form on $T^r M$, uniquely defined by

$$(2.5) \quad d_{[K, L]} = d_K \circ d_L - (-1)^{kl} d_L \circ d_K.$$

The Frölicher-Nijenhuis brackets of the Liouville vector fields \mathbb{C}_α (vector valued 0-forms) and the vertical endomorphisms J^β (vector valued 1-forms) are the vector valued 1-forms given by the following formulae

$$(2.6) \quad [\mathbb{C}_\alpha, J^\beta] = \begin{cases} -\beta J^{\alpha+\beta-1}, & \text{if } \alpha + \beta \leq r+1, \\ 0, & \text{otherwise.} \end{cases}$$

For $r = 1$, semi-basic 1-forms have shown their usefulness to adress various problems associated to second order differential equations fields, [5, 7, 15]. We will see also that for $r > 1$, semi-basic 1-forms, of some order, are useful to formulate a geometric setting for higher order differential equations fields. These forms were introduced and discussed in [3, Def 1]. However, in our work a semi-basic 1-form of order α on $T^r M$ corresponds to a semi-basic 1-form of order $r+1-\alpha$ in [3].

Definition 2.1. A form on $T^r M$ is called *semi-basic of order* $\alpha \in \{1, \dots, r\}$ if it is semi-basic with respect to the submersion $\pi_{\alpha-1}^r$.

A form θ on $T^r M$ is semi-basic of order α if it vanishes whenever one of its argument is a vertical vector field in $\mathfrak{X}^{V_\alpha}(T^r M)$. Therefore, $\theta \in \Lambda^1(T^r M)$ is semi-basic of order α if and only if $i_{J^\alpha} \theta = 0$. Semi-basic 1-forms of order α are the dual equivalent of vertical vector fields in $\mathfrak{X}^{V_\alpha}(T^r M)$. Hence we have that $\theta \in \Lambda^1(T^r M)$ is semi-basic of order α if and only if there exists $\eta \in \Lambda^1(T^r M)$ such that $\theta = i_{J^{r-\alpha+1}} \eta = \eta \circ J^{r-\alpha+1}$. Locally, a 1-form θ on $T^r M$ is semi-basic of order α if and only if

$$(2.7) \quad \theta = \theta_{(0)i} dx^i + \theta_{(1)i} dy^{(1)i} + \dots + \theta_{(\alpha-1)i} dy^{(\alpha-1)i},$$

where α components $\theta_{(0)i}, \dots, \theta_{(\alpha-1)i}$ are smooth functions defined on domains of local charts on $T^r M$.

For a function $f \in C^\infty(T^r M)$ and $\alpha \in \{1, \dots, r\}$ we have that $d_{J^\alpha} f$ is a semi-basic 1-form of order $r - \alpha + 1$. For a function $f \in C^\infty(T^r M)$ we have that df is a semi-basic 1-form of order $\alpha \in \{1, \dots, r\}$ if and only if f is constant along the fibers of the submersion $\pi_{\alpha-1}^r$ and hence one can restrict it to $T^{\alpha-1} M$.

2.2. Higher order differential equations fields. A system of higher order differential equations, whose coefficients do not depend explicitly on time, can be viewed as a special vector field on some higher order tangent bundle. For such systems, we will use the definition for homogeneous differential equations fields of order r , which was proposed by Crampin and Saunders in [14].

As it happens in the case $r = 1$, Liouville vector fields \mathbb{C}_α , are important for defining the notion of homogeneity for various geometric structures on $T^r M$. Whenever we want to consider homogeneous structures, which are not necessarily polynomial in the fibre coordinates, we will consider them defined on the subbundle $T_0^r M$.

Definition 2.2. Consider a vector field S on $T^r M$.

- i) We say that S is a *semispray of order* r if it satisfies the following condition $JS = \mathbb{C}_1$.
- ii) A semispray $S \in \mathfrak{X}(T_0^r M)$ of order r is called *homogeneous* if the distribution $\mathcal{D} = \text{span}\{S, \mathbb{C}_1, \dots, \mathbb{C}_r\}$ is involutive.

In induced coordinates for $T^r M$, a semispray of order r is given by

$$(2.8) \quad S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ry^{(r)i} \frac{\partial}{\partial y^{(r-1)i}} - (r+1)G^i \frac{\partial}{\partial y^{(r)i}},$$

for some functions G^i defined on domains of induced local charts.

Alternatively, we have that a vector field S on $T^r M$ is a semispray of order r if and only if any integral curve of S , $\gamma : I \rightarrow T_0^r M$ is of the form $\gamma = j^r(\pi_0^r \circ \gamma)$. For an integral curve $\gamma : I \rightarrow T_0^r M$ of S , we say that curve $c = \pi_0^r \circ \gamma$ is a *geodesic* of S . Therefore, a regular curve $c : I \rightarrow M$ is a geodesic of S if and only if $S \circ j^r c = (j^r c)'$. Locally, a regular curve $c : I \rightarrow M$, $c(t) = (x^i(t))$, is a geodesic of S if and only if it satisfies the system of $(r+1)$ order ordinary differential equations

$$(2.9) \quad \frac{1}{(r+1)!} \frac{d^{r+1} x^i}{dt^{r+1}} + G^i \left(x, \frac{dx}{dt}, \dots, \frac{1}{r!} \frac{d^r x}{dt^r} \right) = 0.$$

Therefore semisprays of order r describe systems of higher order differential equations which have regular curves on M as solutions.

We will consider also, d_T , the *Tulczyjew differential operator* on $T^r M$, which is given by [34]

$$(2.10) \quad d_T = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ry^{(r)i} \frac{\partial}{\partial y^{(r-1)i}}.$$

Differential operator d_T maps a function $f \in C^\infty(T^\alpha M)$, $\alpha \in \{0, \dots, r-1\}$, into a function $d_T f := d_T(f \circ \pi_\alpha^k)$. The function $d_T f$ is basic with respect to the submersion $\pi_{\alpha+1}^k$, therefore

we can assume that it is defined on $T^{\alpha+1}M$ and hence $d_T f \in C^\infty(T^{\alpha+1}M)$. For an arbitrary semispray of order r , $S \in \mathfrak{X}(T^r M)$, and a function $f \in C^\infty(T^\alpha M)$, $\alpha \in \{0, \dots, r-1\}$, we have also have that $Sf = d_T f \in C^\infty(T^{\alpha+1}M)$.

The Frölicher-Nijenhuis brackets of an arbitrary semispray S and the vertical endomorphisms J^α are useful to fix a (multi) connection on $T^r M$ [3, 4, 12, 29]. In this work we will use only the vertical valued components of these vector valued 1-forms.

Lemma 2.3. *Consider S a semispray of order r .*

i) *The Lie brackets $[\mathbb{C}_\alpha, S]$ are given by*

$$(2.11) \quad [\mathbb{C}_\alpha, S] = \alpha \mathbb{C}_{\alpha-1} + U_\alpha, \quad \forall \alpha \in \{1, \dots, r\},$$

where $\mathbb{C}_0 = S$ and $U_\alpha \in \mathfrak{X}^{V_r}(T^r M)$.

ii) *The vertical components of the Frölicher-Nijenhuis brackets $[S, J^\beta]$ are given by*

$$(2.12) \quad J^\alpha[S, J^\beta] = \begin{cases} -\beta J^{\alpha+\beta-1}, & \text{if } \alpha + \beta \leq r+1, \\ 0, & \text{otherwise.} \end{cases}$$

iii) *For a semi-basic 1-form $\theta \in \Lambda^1(T^r M)$, of order $\alpha \in \{1, \dots, r\}$, we have*

$$(2.13) \quad i_{[S, J^\beta]} \theta = -\beta i_{J^{\beta-1}} \theta, \quad \forall \beta \in \{1, \dots, r\}.$$

iv) *Consider $\theta \in \Lambda^1(T^r M)$ a semi-basic 1-form of order $\alpha \in \{1, \dots, r\}$. Then $\mathcal{L}_{\mathbb{C}_\beta} \theta$ is also a semi-basic 1-form of order α , for all $\beta \in \{1, \dots, r\}$.*

v) *Consider $\theta \in \Lambda^1(T^r M)$ a semi-basic 1-form of order $\alpha \in \{1, \dots, r\}$ such that $\mathcal{L}_S \theta - df$ is a semi-basic 1-form of order 1, for some function f on $T^r M$. Then the function f can be restricted to $T^\alpha M$ and the 1-form θ satisfies the following formulae*

$$(2.14) \quad i_{J^\gamma} \theta = \gamma! \sum_{\beta=1}^{\alpha-\gamma} \frac{(-1)^{\beta-1}}{(\beta+\gamma)!} \mathcal{L}_S^{\beta-1} d_{J^{\beta+\gamma}} f, \quad \forall \gamma \in \{0, 1, \dots, \alpha-1\}.$$

Proof. For $\beta \in \{1, \dots, r\}$ and for every $X \in \mathfrak{X}(T^r M)$ we have

$$(2.15) \quad [S, J^\beta X] - J^\beta[S, X] + \beta J^{\beta-1} X = -U_\beta \in \text{Ker } J = \text{Im } J^r,$$

which has been shown in [4, (3.27)].

i) In formula (2.15) we take $X = S$ and use $J^\beta S = \mathbb{C}_\beta$, for all $\beta \in \{1, \dots, r\}$. It follows formula (2.11).

ii) By composing in formula (2.15) to the left with J^α , for $\alpha \geq 1$, we obtain formula (2.12).

iii) Consider $\theta \in \Lambda^1(T^r M)$ a semi-basic 1-form of order α . It follows that there exists $\eta \in \Lambda^1(T^r M)$ such that $\theta = i_{J^{r-\alpha+1}} \eta$. Using now formula (2.12) we obtain

$$i_{[S, J^\beta]} \theta = i_{J^{r-\alpha+1}[S, J^\beta]} \eta = -\beta i_{J^{r-\alpha+\beta}} \eta = -\beta i_{J^{\beta-1}} \theta,$$

which shows that formula (2.13) is true.

iv) Since $\theta \in \Lambda^1(T^r M)$ is a semi-basic 1-form of order $\alpha \in \{1, \dots, r\}$, it follows that $i_{J^\alpha} \theta = 0$. Using the corresponding commutation rules and formulae (2.6) it follows

$$i_{J^\alpha} \mathcal{L}_{\mathbb{C}_\beta} \theta = \mathcal{L}_{\mathbb{C}_\beta} i_{J^\alpha} \theta + i_{[J^\alpha, \mathbb{C}_\beta]} \theta = \alpha i_{J^{\alpha+\beta-1}} \theta = 0,$$

which proves that $\mathcal{L}_{\mathbb{C}_\beta} \theta$ is a semi-basic 1-form of order α .

v) We know that $\theta \in \Lambda^1(T^r M)$ is a semi-basic 1-form of order α , which means $i_{J^\alpha} \theta = 0$. Moreover, there exists a function $f \in C^\infty(T^r M)$ such that $\mathcal{L}_S \theta - df$ is a semi-basic 1-form of order 1, which means that we have

$$(2.16) \quad i_J \mathcal{L}_S \theta = d_J f.$$

If we apply $i_{J^{\alpha-1}}$ to both sides of formula (2.16) and use the commutation rule we obtain

$$(2.17) \quad \mathcal{L}_S i_{J^\alpha} \theta + i_{[J^\alpha, S]} \theta = df \circ J^\alpha.$$

Using formula (2.13) it follows that $i_{[J^\alpha, S]} \theta = \alpha i_{J^{\alpha-1}} \theta$. From formula (2.17) we have that $\alpha i_{J^{\alpha-1}} \theta = i_{J^\alpha} df$. Consequently, we have that $df \circ J^{\alpha+1} = 0$, which means that $df \in \Lambda^1(T^r M)$ is a semi-basic 1-form of order $\alpha + 1$. It follows that f is constant on the fibres of $\pi_\alpha^r : T^r M \rightarrow T^\alpha M$ and therefore, we can restrict f to $T^\alpha M$ and assume that it is a function defined on $T^\alpha M$.

We will prove now that θ satisfies formulae (2.14). We have seen above that

$$(2.18) \quad i_{J^{\alpha-1}} \theta = \frac{1}{\alpha} d_{J^\alpha} f,$$

which is formula (2.14) for $\gamma = \alpha - 1$. We apply \mathcal{L}_S to both sides of this formula, use the commutation rule, and obtain

$$i_{J^{\alpha-1}} \mathcal{L}_S \theta + i_{[S, J^{\alpha-1}]} \theta = \frac{1}{\alpha} \mathcal{L}_S d_{J^\alpha} f.$$

We use now formulae (2.16) and (2.13) to obtain

$$d_{J^{\alpha-1}} f - (\alpha - 1) i_{J^{\alpha-2}} \theta = \frac{1}{\alpha} \mathcal{L}_S d_{J^\alpha} f.$$

Above formula implies

$$(2.19) \quad i_{J^{\alpha-2}} \theta = \frac{1}{\alpha - 1} d_{J^{\alpha-1}} f - \frac{1}{\alpha(\alpha - 1)} \mathcal{L}_S d_{J^\alpha} f,$$

which is formula (2.14) for $\gamma = \alpha - 2$. We apply again \mathcal{L}_S to both sides of the above formula, use the commutation rule, and obtain

$$(2.20) \quad i_{J^{\alpha-3}} \theta = \frac{1}{\alpha - 2} d_{J^{\alpha-2}} f - \frac{1}{(\alpha - 1)(\alpha - 2)} \mathcal{L}_S d_{J^{\alpha-1}} f + \frac{1}{\alpha(\alpha - 1)(\alpha - 2)} \mathcal{L}_S^2 d_{J^\alpha} f,$$

which is formula (2.14) for $\gamma = \alpha - 3$. We continue the process and obtain

$$(2.21) \quad i_J \theta = \frac{1}{2} d_{J^2} f - \frac{1}{2 \cdot 3} \mathcal{L}_S d_{J^3} f + \cdots + \frac{(-1)^{\alpha-2}}{2 \cdots \alpha} \mathcal{L}_S^{\alpha-2} d_{J^\alpha} f.$$

Formula (2.21) represents formula (2.14) for $\gamma = 1$. Now, for the last step we use above formula, formula (2.13), for $\beta = 1$, as well as formula (2.16):

$$\theta = -i_{[S, J]} \theta = i_J \mathcal{L}_S \theta - \mathcal{L}_S i_J \theta = d_J f - \mathcal{L}_S i_J \theta.$$

It follows that θ is given by formula

$$(2.22) \quad \theta = \sum_{\beta=1}^{\alpha} \frac{(-1)^{\beta-1}}{\beta!} \mathcal{L}_S^{\beta-1} d_{J^\beta} f,$$

which represents formula (2.14) for $\gamma = 0$. □

Formula (2.12) was proven for the case $\alpha = \beta = 1$ in [12, Lemma 5.1].

We pay more attention and discuss now the homogeneity condition ii) of Definition 2.2. In view of formula (2.11), a semispray $S \in \mathfrak{X}(T_0^r M)$ of order r , is homogeneous if and only if for the vertical vector fields $U_\alpha \in \mathfrak{X}^{V_r}(T_0^r M)$, there exists functions $P_\alpha \in C^\infty(T_0^r M)$ such that $U_\alpha = P_\alpha \mathbb{C}_r$, for all $\alpha \in \{1, \dots, r\}$. Therefore, a semispray S of order r is homogeneous if and only if there exists functions $P_\alpha \in C^\infty(T_0^r M)$, $\alpha \in \{1, \dots, r\}$, such that

$$(2.23) \quad [\mathbb{C}_1, S] = S + P_1 \mathbb{C}_r, \quad [\mathbb{C}_\alpha, S] = \alpha \mathbb{C}_{\alpha-1} + P_\alpha \mathbb{C}_r, \quad \alpha \in \{1, \dots, r\}.$$

If we write the Jacobi identities for the vector fields $S, \mathbb{C}_1, \dots, \mathbb{C}_r$, and use the above formulae, we obtain that functions P_1, \dots, P_r must satisfy some consistency conditions. Formulae (2.23) and the consistency conditions for functions P_1, \dots, P_r were obtained in [14, Prop. 4].

For homogeneous higher order differential equations fields, an important concept is that of projective equivalence, which we borrow from [14, Def. 2].

Definition 2.4. Consider S_1 and S_2 two homogeneous semisprays of order r . We say that S_1 and S_2 are *projectively equivalent* if there exists a function $P \in C^\infty(T_0^r M)$ such that $S_1 = S_2 - (r + 1)P\mathbb{C}_r$.

Two homogeneous semisprays S_1 and S_2 , locally given by formula (2.8), are projectively equivalent if and only if the semispray coefficients G_1^i and G_2^i are related by $G_1^i = G_2^i + Py^{(1)i}$, for some function $P \in C^\infty(T_0^r M)$.

Definition 2.5. A homogeneous semispray $S \in \mathfrak{X}(T_0^r M)$ is called a *spray of order r* if $[\mathbb{C}_1, S] = S$ and $[\mathbb{C}_2, S] = 2\mathbb{C}_1$.

Above definition was proposed in [14] for *generalized sprays* and it is motivated by the following arguments. It has been shown in [14, Thm. 4] that for two projectively equivalent homogeneous semisprays their geodesics coincide up to an orientation preserving reparameterization. Moreover, according to [14, Thm. 4], the projective class of a homogeneous semispray contains a spray, that is a homogenous semispray for which the homogeneity conditions (2.23) hold true with $P_1 = P_2 = 0$.

For $r = 1$, a very important property of a spray S is that $[J, S]S = S$, which means that S is an eigenvector of the induced almost product structure, with the corresponding eigenvalue $+1$. Definition 2.2 and formula (2.15) imply a similar property $[J, S]S = S$ for a spray of order r .

3. THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS FOR HIGHER ORDER DIFFERENTIAL EQUATIONS FIELDS

The inverse problem of the calculus of variations for a semispray (of order 1) was reformulated in [5] in terms of semi-basic 1-forms. In this section we extend these aspects to the higher order case. In Theorem 3.4 we characterize Lagrangian semisprays of order $2k - 1$ in terms of semi-basic 1-forms of order k .

3.1. Higher order Lagrangians. In this subsection we discuss some aspects regarding the geometry of a Lagrangian of order k . In Lemma 3.2 we study these geometric aspects in connection with the Poincaré-Cartan 1-form, which is a semi-basic 1-form of order k .

Definition 3.1. A function $L \in C^\infty(T^k M)$ is called a *Lagrangian of order k* if the (0,2)-type tensor, with components

$$(3.1) \quad g_{ij} = \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}},$$

has rank at least 1 on $T^k M$. We say that a Lagrangian L is *regular* if the tensor (3.1) has rank n on $T^k M$.

Consider a Lagrangian $L \in C^\infty(T^k M)$. The *Poincaré-Cartan 1-form* $\theta_L \in \Lambda^1(T^{2k-1} M)$ is given by

$$(3.2) \quad \theta_L = \sum_{\alpha=1}^k \frac{(-1)^{\alpha-1}}{\alpha!} \mathcal{L}_S^{\alpha-1} d_{J^\alpha} L,$$

where $S \in \mathfrak{X}(T^{2k-1} M)$ is an arbitrary semispray of order $2k - 1$. We will see in Lemma 3.2 that θ_L does not depend on S . The *Poincaré-Cartan 2-form* $\omega_L \in \Lambda^2(T^{2k-1} M)$ is given by $\omega_L = -d\theta_L$.

The energy Lagrangian function $\mathcal{E}_L \in C^\infty(T^{2k-1}M)$ is given by

$$(3.3) \quad \mathcal{E}_L = \sum_{\alpha=1}^k \frac{(-1)^{\alpha-1}}{\alpha!} \mathcal{L}_S^{\alpha-1} \mathbb{C}_\alpha(L) - L.$$

In the next Lemma we discuss some geometric aspects for a Lagrangian L of order k in terms of its Poincaré-Cartan forms and the energy Lagrangian function.

Lemma 3.2. *Consider L a Lagrangian on $T^k M$.*

- i) *The Poincaré-Cartan 1-form θ_L is a non-zero, semi-basic 1-form of order k on $T^{2k-1}M$, which does not depend on the semispray S .*
- ii) *The energy Lagrangian function $\mathcal{E}_L \in C^\infty(T^{2k-1}M)$ does not depend on the semispray S and it is related to the Poincaré-Cartan 1-form θ_L by the following formula*

$$(3.4) \quad \mathcal{E}_L = i_S \theta_L - L.$$

- iii) *The Poincaré-Cartan 1-form satisfies $\text{rank}(d\theta_L) = 2k \cdot \text{rank}(g_{ij})$. Therefore, L is a regular Lagrangian if and only if the Poincaré-Cartan 2-form ω_L is a symplectic 2-form on $T^{2k-1}M$.*

Proof. i) Locally, the Poincaré-Cartan 1-form θ_L can be expressed as follows

$$(3.5) \quad \theta_L = \theta_{(0)i} dx^i + \cdots + \theta_{(k-1)i} dy^{(k-1)i},$$

where

$$(3.6) \quad \begin{aligned} \theta_{(0)i} &= \frac{1}{1!} \frac{\partial L}{\partial y^{(1)i}} - \frac{1}{2!} \mathcal{L}_S \left(\frac{\partial L}{\partial y^{(2)i}} \right) + \cdots + \frac{(-1)^{k-1}}{k!} \mathcal{L}_S^{k-1} \left(\frac{\partial L}{\partial y^{(k)i}} \right), \\ \theta_{(1)i} &= \frac{1}{2} \frac{\partial L}{\partial y^{(1)i}} - \frac{1}{2 \cdot 3} \mathcal{L}_S \left(\frac{\partial L}{\partial y^{(2)i}} \right) + \cdots + \frac{(-1)^{k-2}}{2 \cdot 3 \cdots k} \mathcal{L}_S^{k-2} \left(\frac{\partial L}{\partial y^{(k)i}} \right), \\ &\dots \\ \theta_{(k-2)i} &= \frac{1}{k-1} \frac{\partial L}{\partial y^{(k-1)i}} - \frac{1}{k(k-1)} \mathcal{L}_S \left(\frac{\partial L}{\partial y^{(k)i}} \right), \\ \theta_{(k-1)i} &= \frac{1}{k} \frac{\partial L}{\partial y^{(k)i}}. \end{aligned}$$

Consider d_T , the Tulczyjew operator on $T^{2k-1}M$. Since L is a Lagrangian on $T^k M$, it follows that $\partial L / \partial y^{(\alpha)i}$ are locally defined on $T^k M$, for all $\alpha \in \{1, \dots, k\}$. Therefore, we can view

$$(3.7) \quad \mathcal{L}_S^\beta \left(\frac{\partial L}{\partial y^{(\alpha)i}} \right) = d_T^\beta \left(\frac{\partial L}{\partial y^{(\alpha)i}} \right),$$

as locally defined functions on $T^{k+\beta}M$, for all $\beta \in \{1, \dots, k-1\}$. It follows that all components $\theta_{(\alpha)i}$, $\alpha \in \{0, \dots, k-1\}$, in formula (3.6), do not depend on the semispray S . From formula (3.5) it follows that θ_L is a semi-basic 1-form of order k , which does not depend on the semispray S .

ii) Since for all $\alpha \in \{1, \dots, k\}$ the functions $\mathbb{C}_\alpha(L)$ are defined on $T^k M$, it follows that we can view the functions

$$\mathcal{L}_S^{\alpha-1} \mathbb{C}_\alpha(L) = d_T^{\alpha-1} \mathbb{C}_\alpha(L)$$

as being defined on $T^{k+\alpha-1}M$. Therefore, the right hand side of formula (3.3), and hence the energy \mathcal{E}_L , is independent of the choice of the semispray S .

If we apply i_S to both sides of formula (3.2) it follows

$$\begin{aligned}
i_S \theta_L &= \sum_{\alpha=1}^k \frac{(-1)^{\alpha-1}}{\alpha!} i_S \mathcal{L}_S^{\alpha-1} d_{J^\alpha} L = \sum_{\alpha=1}^k \frac{(-1)^{\alpha-1}}{\alpha!} \mathcal{L}_S^{\alpha-1} i_S d_{J^\alpha} L \\
(3.8) \quad &= \sum_{\alpha=1}^k \frac{(-1)^{\alpha-1}}{\alpha!} \mathcal{L}_S^{\alpha-1} \mathcal{L}_{J^\alpha S} L = \sum_{\alpha=1}^k \frac{(-1)^{\alpha-1}}{\alpha!} \mathcal{L}_S^{\alpha-1} \mathbb{C}_\alpha L.
\end{aligned}$$

In the above formula we did use the commutation rule $i_S d_{J^\alpha} + d_{J^\alpha} i_S = \mathcal{L}_{J^\alpha S} + i_{[J^\alpha, S]}$, [15, A.1], as well as the fact that $J^\alpha S = C_\alpha$. From formula (3.8) we obtain that (3.4) is true.

iii) Using formula (3.5) and the fact that we can view $\theta_{(k-\alpha)i}$ as locally defined functions on $T^{(k+\alpha-1)}M$, it follows that

$$(3.9) \quad \text{rank}(d\theta_L) = 2 \cdot \sum_{\alpha=1}^k \text{rank} \left(\frac{\partial \theta_{(k-\alpha)i}}{\partial y^{(k+\alpha-1)j}} \right).$$

Since $\partial L / \partial y^{(k)i}$ are locally defined functions on $T^k M$, we have

$$(3.10) \quad \mathcal{L}_S^\alpha \left(\frac{\partial L}{\partial y^{(k)i}} \right) = (k+1) \cdots (k+\alpha) y^{(k+\alpha)j} g_{ij} + f_\alpha,$$

for f_α locally defined functions on $T^{k+\alpha-1}M$. Using the formulae (3.7) and (3.10) and the components $\theta_{(\alpha)i}$ of the Poincaré-Cartan 1-form θ_L it follows

$$(3.11) \quad \frac{\partial \theta_{(k-\alpha)i}}{\partial y^{(k+\alpha-1)j}} = (-1)^{\alpha-1} \frac{(k+\alpha-1)!(k-\alpha)!}{(k!)^2} g_{ij}, \forall \alpha \in \{1, \dots, k\}.$$

Now, from formulae (3.9) and (3.11) it follows that $\text{rank}(d\theta_L) = 2k \cdot \text{rank}(g_{ij})$. \square

The components $\theta_{(\alpha)i}$, $\alpha \in \{0, \dots, k-1\}$, of the Poincaré-Cartan 1-form θ_L , in formula (3.6), are the Jacobi-Ostrogradski generalized momenta, [18].

The local expression (3.5) - (3.6) for θ_L can be written in a more compact form as follows

$$(3.12) \quad \theta_L = \sum_{\alpha=1}^k (\alpha-1)! \left\{ \sum_{\beta=\alpha}^k \frac{(-1)^{\beta-\alpha}}{\beta!} \mathcal{L}_S^{\beta-\alpha} \left(\frac{\partial L}{\partial y^{(\beta)i}} \right) \right\} dy^{(\alpha-1)i}.$$

3.2. Lagrangian semisprays. The inverse problem of the calculus of variations for systems of higher order ordinary differential equations can be formulated as follows. Under what conditions the solutions of the system (2.9) of order $r = 2k$ are also solutions of the Euler-Lagrange equations

$$(3.13) \quad \frac{\partial L}{\partial x^i} - \frac{1}{1!} \frac{d}{dt} \left(\frac{\partial L}{\partial y^{(1)i}} \right) + \cdots + \frac{(-1)^k}{k!} \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial y^{(k)i}} \right) = 0,$$

for some Lagrangian $L \in C^\infty(T^k M)$?

Definition 3.3. A semispray S , of order $2k-1$, is called a (locally defined) *Lagrangian semispray* if its geodesics, which are solutions to the system (2.9) for $r = 2k$, are solutions to the Euler-Lagrange equations (3.13), for some (locally defined) Lagrangian L of order k .

Next theorem provides a characterization for Lagrangian semisprays, in terms of semi-basic 1-forms, extending the results obtain in [5]. In [20, Thm. 3.2], Lagrangian semisprays of order $2k$ are characterized in terms of a closed 2-form, extending the $k = 1$ case, which was studied in [10].

Theorem 3.4. Consider S a semispray of order $2k-1$.

- i) S is a Lagrangian semispray if and only if there exists a Lagrangian L of order k such that either one, of the following equivalent two conditions, is satisfied

$$(3.14) \quad \mathcal{L}_S \theta_L = dL, \quad i_S \omega_L = d\mathcal{E}_L.$$

- ii) S is a (locally defined) Lagrangian semispray if and only if there exists a semi-basic 1-form $\theta \in \Lambda^1(T^{2k-1}M)$ of order k such that $\text{rank}(d\theta) \geq 2k$ and the 1-form $\mathcal{L}_S \theta$ is closed. In this case θ is the Poincaré-Cartan 1-form of some locally defined Lagrangian L of order k .

Proof. i) Using the Euler-Lagrange equations (3.13), it follows that the semispray S is Lagrangian if and only if it satisfies the equation

$$(3.15) \quad \frac{\partial L}{\partial x^i} - \frac{1}{1!} S \left(\frac{\partial L}{\partial y^{(1)i}} \right) + \cdots + \frac{(-1)^k}{k!} S^k \left(\frac{\partial L}{\partial y^{(k)i}} \right) = 0,$$

for some Lagrangian function L on $T^k M$.

In view of formula (3.4) we obtain that the two equations (3.14) are equivalent. Therefore, we will have to prove that equation (3.15) and first equation (3.14) are equivalent.

Using expression (3.12) for the Poincaré-Cartan 1-form θ_L and the fact that $\mathcal{L}_S dy^{(\alpha-1)i} = \alpha dy^{(\alpha)i}$ it follows

$$\begin{aligned} \mathcal{L}_S \theta_L &= \sum_{\alpha=1}^k (\alpha-1)! \left\{ \sum_{\beta=\alpha}^k \frac{(-1)^{\beta-\alpha}}{\beta!} \mathcal{L}_S^{\beta-\alpha+1} \left(\frac{\partial L}{\partial y^{(\beta)i}} \right) \right\} dy^{(\alpha-1)i} \\ &\quad + \sum_{\alpha=1}^k \alpha! \left\{ \sum_{\beta=\alpha}^k \frac{(-1)^{\beta-\alpha}}{\beta!} \mathcal{L}_S^{\beta-\alpha} \left(\frac{\partial L}{\partial y^{(\beta)i}} \right) \right\} dy^{(\alpha)i} \\ &= \sum_{\beta=1}^k \frac{(-1)^{\beta-1}}{\beta!} \mathcal{L}_S^\beta \left(\frac{\partial L}{\partial y^{(\beta)i}} \right) dx^i + \sum_{\alpha=1}^k \frac{\partial L}{\partial y^{(\alpha)i}} dy^{(\alpha)i}. \end{aligned}$$

If we use the above expression for $\mathcal{L}_S \theta_L$ it follows that

$$(3.16) \quad dL - \mathcal{L}_S \theta_L = \left\{ \frac{\partial L}{\partial x^i} + \sum_{\beta=1}^k \frac{(-1)^\beta}{\beta!} \mathcal{L}_S^\beta \left(\frac{\partial L}{\partial y^{(\beta)i}} \right) \right\} dx^i$$

is a semi-basic 1-form on $T^{2k-1}M$ of order 1. Formula (3.16) shows that equation (3.15) and first equation (3.14) are equivalent.

ii) For the direct implication of this part, we assume that S is a Lagrangian semispray. Therefore, semispray S satisfies first equation (3.14), for some Lagrangian L of order k . We consider $\theta = \theta_L \in \Lambda^{2k-1}(T^k M)$, its Poincaré-Cartan 1-form, which is a semi-basic 1-form of order k and satisfies first equation (3.14). Using last part of Lemma 3.2 it follows that $\text{rank}(d\theta) = 2k \cdot \text{rank}(g_{ij}) \geq 2k$.

For the converse, let us consider $\theta \in \Lambda^{2k-1}(T^k M)$, a semi-basic 1-form of order k such that $\mathcal{L}_S \theta$ is a closed 1-form. Therefore $\mathcal{L}_S \theta$ is locally exact and hence there exists L , a locally defined function on $T^{2k-1}M$, such that

$$(3.17) \quad \mathcal{L}_S \theta = dL.$$

We want to prove now that L is constant on the fibres π_k^{2k-1} and hence we can view it as a function on $T^k M$. Moreover, we will prove that θ is the Poincaré-Cartan 1-form θ_L of L . For these, as we have seen in the last part of Lemma 2.3, we need a condition weaker than (3.17), namely we will use the fact that $\mathcal{L}_S \theta - dL$ is a semi-basic 1-form of order 1. This means that

$$i_J \mathcal{L}_S \theta = d_J L.$$

According to part v) of Lemma 2.3 it follows that one can restrict the function L to $T^k M$ and the semi-basic 1-form θ is given by formula (2.22), where $f = L$ and $\alpha = k$. It follows that θ is given by formula (3.2) and hence it is the Poincaré-Cartan 1-form of the function $L \in C^\infty(T^k M)$. Using third part of Lemma 3.2 it follows the metric tensor g_{ij} of L , given by formula (3.1), satisfies $2k \cdot \text{rank}(g_{ij}) = \text{rank} d\theta \geq 2k$. This means that $\text{rank}(g_{ij}) \geq 1$ and hence L is a (locally defined) Lagrangian function on $T^k M$. If we replace $\theta = \theta_L$ in formula (3.17) it follows that the semispray S satisfies first formula (3.14) for the Lagrangian L . In view of the first part of the theorem it follows that the semispray S is Lagrangian. \square

According to last part of Lemma 3.2, we have that for a regular Lagrangian L , second equation (3.14) has a unique solution. This way, to each regular Lagrangian L on $T^k M$ it corresponds a unique Lagrangian semispray $S \in \mathfrak{X}(T^{2k-1} M)$. Using the terminology introduced in [17, Ch. 4] we can say that for a regular Lagrangian, its Euler-Lagrange distribution, which is spanned by the semispray S , has a constant rank equal to one.

If we want to determine the local coefficients G^i of a Lagrangian semispray S of order $2k - 1$, we use formula (3.10) and write equations (3.15) in the following equivalent form

$$(3.18) \quad (-1)^k \binom{2k}{k} g_{ij} G^j = \frac{\partial L}{\partial x^i} - \frac{1}{1!} d_T \left(\frac{\partial L}{\partial y^{(1)i}} \right) + \cdots + \frac{(-1)^k}{k!} d_T^k \left(\frac{\partial L}{\partial y^{(k)i}} \right).$$

It follows that for a regular Lagrangian, the metric tensor g_{ij} is invertible and hence equations (3.18) uniquely determine the semispray coefficients G^i .

For a Lagrangian semispray S , its geodesics are solutions of the Euler-Lagrange equations (3.13). Moreover, the geodesic equations (2.9), with $r = 2k$, for the Lagrangian semispray S and the Euler-Lagrange equations (3.13) are related by

$$(3.19) \quad \frac{(-1)^k}{k!} g_{ij} \left\{ \frac{d^{2k} x^j}{dt^{2k}} + (2k)! G^j \right\} = \frac{\partial L}{\partial x^i} - \frac{1}{1!} \frac{d}{dt} \left(\frac{\partial L}{\partial y^{(1)i}} \right) + \cdots + \frac{(-1)^k}{k!} \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial y^{(k)i}} \right)$$

where g_{ij} is the tensor (3.1). The two systems of equations (2.9) and (3.13) coincide if the Lagrangian is regular.

Next lemma presents some compatibility conditions between the geometric structures associated to a Lagrangian and the Liouville vector fields. These properties will be useful in the next section to see how the homogeneity properties of a Finsler function transfer to the induced geometric structures.

Lemma 3.5. *Consider L a Lagrangian on $T^k M$ and $\theta_L \in \Lambda^1(T^{2k-1} M)$ its Poincaré-Cartan 1-form. The following formulae are true:*

$$(3.20) \quad \begin{aligned} \mathcal{L}_{\mathbb{C}_1} \theta_L &= \theta_{\mathbb{C}_1(L)-L}, \\ i_{\mathbb{C}_\alpha} \theta_L &= \alpha! \sum_{\beta=1}^{k-\alpha} \frac{(-1)^{\beta-1}}{(\alpha+\beta)!} \mathcal{L}_S^{\beta-1} (\mathbb{C}_{\alpha+\beta}(L)), \quad \forall \alpha \in \{1, \dots, k-1\}, \\ i_{\mathbb{C}_\alpha} \theta_L &= 0, \quad \forall \alpha \in \{k, \dots, 2k-1\}. \end{aligned}$$

Proof. For the Lagrangian function L consider S a semispray, solution to one of the two equivalent equations (3.14), which means $\mathcal{L}_S \theta_L = dL$. If we apply $\mathcal{L}_{\mathbb{C}_1}$ to both sides of this formula and use the commutation rule we obtain

$$(3.21) \quad \mathcal{L}_S \mathcal{L}_{\mathbb{C}_1} \theta_L + \mathcal{L}_{[\mathbb{C}_1, S]} \theta_L = d\mathbb{C}_1(L).$$

Using formula (2.11), it follows that $[\mathbb{C}_1, S] = S + U_1$, for $U_1 \in \mathfrak{X}^{V_{2k-1}}(T^{2k-1} M)$. If we replace $[\mathbb{C}_1, S]$ in formula (3.21) we obtain

$$(3.22) \quad \mathcal{L}_S \mathcal{L}_{\mathbb{C}_1} \theta_L = d(\mathbb{C}_1(L) - L) - \mathcal{L}_{U_1} \theta_L.$$

Using the local expression (3.5) of the Poincaré-Cartan 1-form θ_L and the fact that its only component that depends on $y^{(2k-1)i}$ is $\theta_{(0)i}$, which is given in formula (3.6), it follows that

$$\mathcal{L}_{U_1}\theta_L = \mathcal{L}_{U_1}(\theta_{(0)i})dx^i.$$

Therefore $\mathcal{L}_{U_1}\theta_L$ is a semi-basic 1-form of order 1. Using formula (3.22) it follows

$$(3.23) \quad i_J\mathcal{L}_S\mathcal{L}_{\mathbb{C}_1}\theta_L = i_Jd(\mathbb{C}_1(L) - L).$$

According to part iv) of Lemma 2.3 it follows that $\mathcal{L}_{\mathbb{C}_1}\theta_L$ is a semi-basic 1-form of order k . We use now part v) of Lemma 2.3 to conclude, from formula (3.23), that the semi-basic 1-form of order k , $\mathcal{L}_{\mathbb{C}_1}\theta_L$, satisfies formula (2.22) for $\alpha = k$ and $f = \mathbb{C}_1(L) - L$. In view of formula (3.2), this means that $\mathcal{L}_{\mathbb{C}_1}\theta_L$ is the Poincaré-Cartan 1-form of the function $\mathbb{C}_1(L) - L$, which is first formula (3.20).

Now, we use formula $\mathcal{L}_S\theta_L = dL$ and compose both sides with i_J , which means that $i_J\mathcal{L}_S\theta_L = d_JL$. Using this formula and part v) of Lemma 2.3 it follows that the semi-basic 1-form of order k , θ_L satisfies formulae (2.14) for $\alpha = k$, $\gamma \in \{0, \dots, k-1\}$ and $f = L$, which can be written as follows

$$(3.24) \quad i_{J^\alpha}\theta_L = \alpha! \sum_{\beta=1}^{k-\alpha} \frac{(-1)^{\beta-1}}{(\alpha+\beta)!} \mathcal{L}_S^{\beta-1} d_{J^{\alpha+\beta}}L, \quad \forall \alpha \in \{1, \dots, k-1\}.$$

We note that both sides in above formulae do not depend on the chosen semispray S . If we compose with i_S in both sides of formulae (3.24), we obtain formulae (3.20) for $\alpha \in \{1, \dots, k-1\}$.

Since θ_L is a semi-basic 1-form of order k , it follows that there exists $\eta \in \Lambda^1(T^{2k-1}M)$ such that $\theta_L = i_{J^k}\eta$. For $\alpha \in \{k, \dots, 2k-1\}$, we have that $J^k(\mathbb{C}_\alpha) = 0$. Therefore, $i_{\mathbb{C}_\alpha}\theta_L = i_{J^k(\mathbb{C}_\alpha)}\eta = 0$ and hence we proved all formulae (3.20) \square

The 1-forms $i_{J^\alpha}\theta_L \in \Lambda^1(T^{2k-1}M)$, $\alpha \in \{0, \dots, k-1\}$ are semi-basic 1-forms of order $k-\alpha$.

We prove in the next proposition that some homogeneity properties of a regular Lagrangian are inherited by the canonical Euler-Lagrange vector field.

Proposition 3.6. *Consider $L \in C^\infty(T_0^k M)$ a regular Lagrangian of order k such that $\mathbb{C}_1(L) = aL$, and let S be its canonical semispray of order $2k-1$. It follows that $[\mathbb{C}_1, S] = S$.*

Proof. Since L is a regular Lagrangian of order k it follows that the semispray $S \in \mathfrak{X}(T_0^{2k-1}M)$ is the unique solution of the second equation (3.14). Using the fact that $\mathcal{L}_S\omega_L = 0$, it follows that

$$(3.25) \quad i_{[\mathbb{C}_1, S]}\omega_L = i_{\mathbb{C}_1}\mathcal{L}_S\omega_L - \mathcal{L}_Si_{\mathbb{C}_1}\omega_L = \mathcal{L}_Si_{\mathbb{C}_1}d\theta_L = \mathcal{L}_S(\mathcal{L}_{\mathbb{C}_1}\theta_L - di_{\mathbb{C}_1}\theta_L).$$

If we use first formula (3.20) and the homogeneity condition $\mathbb{C}_1(L) = aL$ we obtain $\mathcal{L}_{\mathbb{C}_1}\theta_L = (a-1)\theta_L$. We replace this and first formula (3.14) in (3.25). It follows

$$(3.26) \quad i_{[\mathbb{C}_1, S]}\omega_L = (a-1)\mathcal{L}_S\theta_L - d\mathcal{L}_Si_{\mathbb{C}_1}\theta_L = (a-1)dL - d\mathcal{L}_Si_{\mathbb{C}_1}\theta_L.$$

Using second formula (3.20) for $\alpha = 1$ we obtain the following expression for the energy Lagrangian function \mathcal{E}_L , which is given by formula (3.3)

$$(3.27) \quad \mathcal{E}_L = \mathbb{C}_1(L) - \mathcal{L}_Si_{\mathbb{C}_1}\theta_L - L = (a-1)L - \mathcal{L}_Si_{\mathbb{C}_1}\theta_L.$$

We replace the expression for $\mathcal{L}_Si_{\mathbb{C}_1}\theta_L$ from above formula in (3.26) and obtain

$$i_{[\mathbb{C}_1, S]}\omega_L = d\mathcal{E}_L = i_S\omega_L.$$

Since ω_L is a symplectic structure it follows that $[\mathbb{C}_1, S] = S$. \square

For the case $k = 1$, above formulae show that the homogeneity of a regular Lagrangian transfers to the canonical Euler-Lagrange vector field, which makes it into a spray.

4. PROJECTIVE METRIZABILITY FOR HOMOGENEOUS HIGHER ORDER DIFFERENTIAL EQUATIONS FIELDS

A particular aspect of the inverse problem of the calculus of variations deals with homogeneous systems of differential equations. For $k = 1$, this problem is known as the projective metrization problem, or as the Finslerian version of Hilbert's fourth problem [1, 11, 32]. The most important aspect that is needed to formulate and address the projective metrization problem for $k > 1$ relies on a correct definition of homogeneity for systems of higher order differential equations and corresponding Lagrangians. We believe that such definition of homogeneity is that proposed by Crampin and Saunders in [14], which we use in this paper. In this section we formulate and discuss some aspects regarding the projective metrization problem for the case $k > 1$, extending some results obtained in [5, 7] for $k = 1$.

4.1. Higher order Finsler spaces. For $k = 1$, a Finsler function is characterized by the following important aspect: its variational problem uniquely determines a class of projectively related systems of second order ordinary differential equations. This property is due to the fact that a Finsler function satisfies some homogeneity condition and a regularity condition. Inspired by the work of Crampin and Saunders [14], we propose the following definition for a Finsler function of order $k > 1$.

Definition 4.1. A positive function $F \in C^\infty(T_0^k M)$ is called a *Finsler function of order k* if

i) it satisfies the Zermelo conditions:

$$(4.1) \quad \mathbb{C}_1(F) = F, \quad \mathbb{C}_\alpha(F) = 0, \quad \forall \alpha \in \{2, \dots, k\},$$

ii) the tensor with components

$$(4.2) \quad h_{ij} = F^{2k-1} \frac{\partial^2 F}{\partial y^{(k)i} \partial y^{(k)j}}$$

has rank $n - 1$ on $T_0^k M$.

A Lagrangian L on $T_0^k M$ that satisfies the Zermelo conditions (4.1) in Definition 4.1 is called *parametric Lagrangian* in [14, §4] since the solutions of the corresponding variational problem are invariant under orientation preserving reparameterization.

Definition 4.1 reduces to the classic definition of a Finsler space when $k = 1$, and the tensor (4.2) becomes the angular metric tensor [21, §16].

Using last part of Lemma 3.2, the regularity condition ii) in Definition 4.1 is equivalent to $\text{rank}(d\theta_F) = 2k(n - 1)$, which is the regularity condition for parametric Lagrangians considered by Crampin and Saunders in [14]. From the Zermelo condition $\mathbb{C}_k(F) = 0$ we obtain that $h_{ij}y^{(1)j} = 0$. Therefore, due to the regularity condition ii) in Definition 4.1, which requires that $\text{rank}(h_{ij})$ is maximal, we have that the only solutions of the system $h_{ij}X^j = 0$ is given by $X^j = Py^{(1)j}$, for some function $P \in C^\infty(T_0^{2k-1} M)$.

Definition 4.2. A 1-form $\theta \in \Lambda^1(T_0^{2k-1} M)$ is called *homogeneous* if it satisfies the formulae

$$(4.3) \quad i_{\mathbb{C}_\alpha} \theta = 0, \quad \mathcal{L}_{\mathbb{C}_\alpha} \theta = 0, \quad \forall \alpha \in \{1, \dots, 2k - 1\}.$$

Due to the homogeneity conditions of a Finsler function of order k , the energy function \mathcal{E}_F and the Poincaré-Cartan forms θ_F and $\omega_F = -d\theta_F$ have special properties. These properties are presented in the next lemma.

Lemma 4.3. Consider $F \in C^\infty(T_0^k M)$ a Finsler function of order k .

i) The Poincaré-Cartan 1-form θ_F satisfies the homogeneity conditions (4.3).

ii) *The following formulae are true*

$$(4.4) \quad i_S \theta_F = F, \quad \mathcal{E}_F = 0.$$

$$(4.5) \quad i_{\mathbb{C}_\alpha} \omega_F = 0, \quad \forall \alpha \in \{1, \dots, 2k-1\}.$$

Proof. Using second formulae (3.20) it follows $i_{\mathbb{C}_\alpha} \theta_F = 0$ if and only if $\mathbb{C}_{\alpha+1}(F) = 0$ for all $\alpha \in \{1, \dots, k-1\}$.

Since F satisfies the Zermelo conditions (4.1) it follows $i_{\mathbb{C}_\alpha} \theta_F = 0$ for all $\alpha \in \{1, \dots, k-1\}$. Last formulae (3.20) show that $i_{\mathbb{C}_\alpha} \theta_F = 0$ for all $\alpha \in \{k, \dots, 2k-1\}$.

First formula (3.20) shows that the Zermelo condition $\mathbb{C}_1(F) = F$ implies $\mathcal{L}_{\mathbb{C}_1} \theta_F = 0$.

Consider now $S \in \mathfrak{X}(T_0^{2k-1}M)$ a semispray, solution of the equation $\mathcal{L}_S \theta_F = dF$. For $\alpha \geq 2$, we apply $\mathcal{L}_{\mathbb{C}_\alpha}$ to both sides of the above equations. It follows

$$(4.6) \quad \mathcal{L}_S \mathcal{L}_{\mathbb{C}_\alpha} \theta_F + \mathcal{L}_{[\mathbb{C}_\alpha, S]} \theta_F = d\mathbb{C}_\alpha(F) = 0.$$

Using formula (2.11), for each $\alpha \in \{2, \dots, 2k-1\}$ there exists $U_\alpha \in \mathfrak{X}^{V_{2k-1}}(T_0^{2k-1}M)$ such that $[\mathbb{C}_\alpha, S] = \alpha \mathbb{C}_{\alpha-1} + U_\alpha$. We replace this in formula (4.6) and obtain

$$(4.7) \quad \mathcal{L}_S \mathcal{L}_{\mathbb{C}_\alpha} \theta_F + \alpha \mathcal{L}_{\mathbb{C}_{\alpha-1}} \theta_F = -\mathcal{L}_{U_\alpha} \theta_F, \quad \forall \alpha \in \{2, \dots, 2k-1\}.$$

Using a similar argument that we have used in the proof of Lemma 3.5 it follows that $\mathcal{L}_{U_\alpha} \theta_F$ are semi-basic 1-forms of order 1, for all $\alpha \in \{2, \dots, 2k-1\}$. For $\alpha = 2$, in formula (4.7), it follows that $\mathcal{L}_S \mathcal{L}_{\mathbb{C}_2} \theta_F$ is a semi-basic 1-form of order 1. Item v) of Lemma 2.3 implies $\mathcal{L}_{\mathbb{C}_2} \theta_F = 0$. We continue with $\alpha \in \{3, \dots, 2k-1\}$ in formula (4.7), use a similar argument as above, and obtain $\mathcal{L}_{\mathbb{C}_\alpha} \theta_F = 0$.

For θ_F , the Poincaré-Cartan 1-form of a Finsler function F , given by formula (3.2), we use formula (3.8), as well as the Zermelo conditions (4.1), to obtain $i_S \theta_F = \mathbb{C}_1(F) = F$, which is first formula (4.4). These considerations and formula (3.4) imply that second formula (4.4) is true.

The Poincaré-Cartan 1-form is homogeneous, which means that it satisfies formulae (4.3). The two formulae (4.3) imply that formula (4.5) is true as well. \square

The homogeneity properties of the Poincaré-Cartan forms θ_F and ω_F were proven in a different context in Proposition 6.1 and Theorem 6.4 of [13].

4.2. Higher order projective metrizability. In this subsection we formulate and discuss the projective metrizability problem for homogeneous higher order systems. We show first that the variational problem of a Finsler function of order k uniquely determines a projective class of homogeneous higher order systems. Then, we characterize the metrizability of a homogeneous higher order systems in terms of some homogeneous semi-basic 1-forms.

Definition 4.4. A homogeneous semispray $S \in \mathfrak{X}(T_0^{2k-1}M)$ is said to be *projectively metrizable* if its geodesics coincide with the solutions of the Euler-Lagrange equations of some Finsler function F , up to an orientation preserving reparameterization.

The variational problem for a regular Lagrangian on $T^k M$ uniquely determines a Lagrangian semispray of order $2k-1$. In Theorem 3.4 we give characterizations for a semispray S to be a Lagrangian semispray.

We will see now that in the case of a Finsler function of order k , the variational problem uniquely determine a projective class of sprays. In the next theorem, which represents the homogeneous version on Theorem 3.4, we provide characterizations of projectively metrizable homogeneous semisprays in terms of homogeneous semi-basic 1-forms, extending the case $k=1$ studied in [5, §4.3].

Theorem 4.5. *Consider S a homogeneous semispray of order $2k-1$.*

- i) S is projectively metrizable if and only if it satisfies either one of the following equivalent two equations

$$(4.8) \quad \mathcal{L}_S \theta_F = dF, \quad i_S \omega_F = 0,$$

for some Finsler functions F of order k .

- ii) S is projectively metrizable if and only if there exists a homogeneous semi-basic 1-form $\theta \in \Lambda^1(T^{2k-1}M)$ of order k , such $\text{rank}(d\theta) = 2k(n-1)$ and $\mathcal{L}_S \theta$ is a closed 1-form.

Proof. i) In view of the two formulae (4.4) we have that the two equations (4.8) are equivalent.

For the direct implication, we assume that S is projectively metrizable. Then, the semispray S satisfies the equation

$$(4.9) \quad \frac{\partial F}{\partial x^i} - \frac{1}{1!} S \left(\frac{\partial F}{\partial y^{(1)i}} \right) + \cdots + \frac{(-1)^k}{k!} S^k \left(\frac{\partial F}{\partial y^{(k)i}} \right) = 0,$$

for some Finsler function F on $T^k M$. Using similar arguments as we did use in the proof of Theorem 3.4, it follows that equation (4.9) is equivalent to first equation (4.8).

For the converse implication, consider F a Finsler function of order k . We assume that the semispray S is a solution of the second equation (4.8). Locally, first equation (4.8) is equivalent to

$$(4.10) \quad (-1)^k \binom{2k}{k} h_{ij} G^j = \frac{\partial F}{\partial x^i} - \frac{1}{1!} d_T \left(\frac{\partial F}{\partial y^{(1)i}} \right) + \cdots + \frac{(-1)^k}{k!} d_T^k \left(\frac{\partial F}{\partial y^{(k)i}} \right).$$

It follows that two homogeneous semisprays S_1 and S_2 are solutions of either one of the two equations (4.8) if and only if the semispray coefficients G_1^i and G_2^i satisfy

$$(4.11) \quad h_{ij}(G_1^i - G_2^i) = 0.$$

The regularity condition for the Finsler function F implies that the only solutions of equation (4.11) are given by $G_1^i - G_2^i = P y^{(1)i}$, for some function $P \in C^\infty(T_0^{2k-1}M)$, and hence the two homogeneous semisprays S_1 and S_2 are projectively equivalent. Therefore, equations (4.8) uniquely determine the projective class of a homogeneous semispray S , and this homogeneous semispray is projectively metrizable.

ii) For the first implication we assume that the homogeneous semispray S is projectively metrizable. Therefore, it satisfies first equation (4.8), for some Finsler function F of order k . We consider $\theta = \theta_F$, the Poincaré-Cartan 1-form of F . We have that $\theta \in \Lambda^1(T_0^{2k-1}M)$ is a homogeneous, semi-basic 1-form of order k , $\mathcal{L}_S \theta$ is closed, and $\text{rank}(d\theta) = 2k \cdot \text{rank}(h_{ij}) = 2k(n-1)$.

For the converse implication, consider a homogeneous semi-basic 1-form $\theta \in \Lambda^1(T_0^{2k-1}M)$ of order k , such $\text{rank}(d\theta) = 2k(n-1)$ and $\mathcal{L}_S \theta$ is a closed 1-form.

We will prove first that the condition $\mathcal{L}_S \theta$ is closed implies $\mathcal{L}_S \theta = di_S \theta$. Since S is a homogeneous semispray of order $2k-1$ it follows $[J, S]S = S + P_1 \mathbb{C}_{2k-1}$, for some function $P_1 \in C^\infty(T_0^{2k-1}M)$. Due to the homogeneity of the semi-basic 1-form θ it follows that $\mathcal{L}_{P_1 \mathbb{C}_{2k-1}} \theta = 0$. Using the commutation rule [15, 1a, p.205] it follows

$$(4.12) \quad \mathcal{L}_S \theta = \mathcal{L}_{[J,S]} \theta = i_S d_{[J,S]} \theta + d_{[J,S]} i_S \theta + i_{[[J,S],S]} \theta.$$

Since θ is a semi-basic 1-form of order k it follows that for $k \geq 2$ the 1-form $\mathcal{L}_S \theta$ is semi-basic of order $k+1 \leq 2k-1$. Using formula (2.13) for $\beta = 1$ and the commutation rule [15, 1c, p.205] it follows

$$(4.13) \quad i_{[[J,S],S]} \theta = i_{[J,S]} \mathcal{L}_S \theta - \mathcal{L}_S i_{[J,S]} \theta = \mathcal{L}_S \theta - \mathcal{L}_S \theta = 0.$$

Formulae (2.4), (2.13) for $\beta = 1$ and the condition $\mathcal{L}_S d\theta = 0$ imply:

$$d_{[J,S]} \theta = i_{[J,S]} d\theta - di_{[J,S]} \theta = i_{[J,S]} d\theta - d\theta = i_J \mathcal{L}_S d\theta - \mathcal{L}_S i_J d\theta - d\theta = -\mathcal{L}_S i_J d\theta - d\theta.$$

In the above formula we compose with i_S , use the homogeneity condition $i_{\mathbb{C}_1} d\theta = 0$ and obtain

$$\begin{aligned} i_S d_{[J,S]} \theta &= -\mathcal{L}_S i_S i_J d\theta - i_S d\theta = -\mathcal{L}_S (i_J i_S d\theta + i_{\mathbb{C}_1} d\theta) - i_S d\theta = -\mathcal{L}_S i_J i_S d\theta - i_S d\theta \\ (4.14) \quad &= -i_J \mathcal{L}_S i_S d\theta + i_{[J,S]} i_S d\theta - i_S d\theta = i_{[J,S]} i_S d\theta - i_S d\theta. \end{aligned}$$

If we replace now formulae (4.13) and (4.14) and the fact that $d_{[J,S]} i_S \theta = i_{[J,S]} di_S \theta$ in formula (4.12) we obtain

$$\mathcal{L}_S \theta = -i_S d\theta + i_{[J,S]} i_S d\theta + i_{[J,S]} di_S \theta = -i_S d\theta + i_{[J,S]} \mathcal{L}_S \theta = -i_S d\theta + \mathcal{L}_S \theta = di_S \theta.$$

Consider the function $F = i_S \theta$. Above formula shows that the function F satisfies formula (3.17), for $L = F$, which means $\mathcal{L}_S \theta = dF$. Since $\mathcal{L}_S \theta \in \Lambda^1(T_0^{2k-1}M)$ is semi-basic of order $(k+1)$, it follows that the function F is constant along the fibres of the projection $\pi_k^{2k-1} : T_0^{2k-1}M \rightarrow T_0^k M$ and hence we can assume that $F \in C^\infty(T_0^k M)$. Using part v) of Lemma 2.3 we obtain that $\theta = \theta_F$.

We have to prove now that the function F is a Finsler function. From first formulae (3.20) it follows that $i_{\mathbb{C}_\alpha} \theta_F = 0$ if and only if $\mathbb{C}_{\alpha+1}(F) = 0$ for all $\alpha \in \{1, \dots, k-1\}$. Since θ is homogeneous, we obtain $\mathbb{C}_2(F) = \dots = \mathbb{C}_k(F) = 0$. These arguments, the definition of function F and formula (3.8) imply $F = i_S \theta_F = \mathbb{C}_1(F)$. It follows that Zermelo conditions (4.1) are satisfied. Finally, we have that $\text{rank}(d\theta) = 2k(n-1)$, which implies that F is a Finsler function of order k . Now the condition $\mathcal{L}_S \theta_F = dF$ says that S is projectively metrizable. \square

For $k = 1$, second equation (4.8) reduces to Rapcsák equation [32, Rap 1].

We note that for the converse implication of the first part of Theorem 4.5 we do not need the requirement that the semispray S is homogeneous. The argument is as follows, and it is due to Crampin and Saunders [14, Prop. 5]. For a semispray S , solution of second equation (4.8), using formulae (4.5) it follows that $\mathcal{D} = \text{span}\{S, \mathbb{C}_1, \dots, \mathbb{C}_{2k-1}\} = \text{Ker } \omega_F$. Since the Poincaré-Cartan 2-form $\omega_F = -d\theta_F$ is closed it follows that its characteristic distribution \mathcal{D} is involutive and hence S is a homogeneous semispray.

5. EXAMPLES

For a Finsler function F , of order $k \geq 1$, its variational problem uniquely determines a projective class of homogeneous semisprays, solutions of either one of the two equivalent equations (4.8).

For $k = 1$, in this projective class of homogeneous semisprays, we can single out one spray, which is called the geodesic spray. The geodesic spray is the only semispray determined by the variational problem of the regular Lagrangian $L = F^2$. Moreover, the geodesic spray is the only spray, in the projective class determined by the Finsler function F , whose geodesics are parameterized by arc-length.

When $k > 1$, we do not know yet if it is possible, and eventually how, to associate to a Finsler function F of order k , a regular Lagrangian of order k . Therefore, the only option to fix a homogeneous semispray, which was suggested to me by David Saunders, in the projective class determined by the variational problem of F , is to use the arc-length induced by F .

Next we use some examples to discuss the above considerations as well as the results obtained in the previous sections.

For a Riemannian metric $g_{ij}(x)$ on a manifold M , consider the functions $L_1, F_1 : TM \rightarrow \mathbb{R}$, given by

$$(5.1) L_1(x, y^{(1)}) = \frac{1}{2} g_{ij}(x) y^{(1)i} y^{(1)j} = \frac{1}{2} \|y^{(1)}\|_g^2, \quad F_1(x, y^{(1)}) = \sqrt{g_{ij}(x) y^{(1)i} y^{(1)j}} = \|y^{(1)}\|_g.$$

L_1 is a regular Lagrangian of order one, its metric tensor, given by formula (3.1), is just the Riemannian metric $g_{ij}(x)$. The variational problem for L_1 uniquely determines a spray $S_1 \in$

$\mathfrak{X}(TM)$, which is the geodesic spray for the Riemannian metric $g_{ij}(x)$. The geodesic spray S_1 is uniquely determined by either one of the two equations (3.14), for $k = 1$, and it is given by

$$(5.2) \quad S_1 = y^{(1)i} \frac{\partial}{\partial x^i} - \gamma_{jk}^i(x) y^{(1)j} y^{(1)k} \frac{\partial}{\partial y^{(1)i}},$$

where $\gamma_{jk}^i(x)$ are the Christoffel symbols of the Riemannian metric $g_{ij}(x)$.

F_1 is a Finsler function of order one, its angular metric tensor, given by formula (4.2), is related to the Riemannian metric as follows

$$(5.3) \quad h_{ij}^{(1)}(x, y^{(1)}) = F_1 \frac{\partial^2 F_1}{\partial y^{(1)i} \partial y^{(1)j}} = g_{ij}(x) - \frac{\partial F_1}{\partial y^{(1)i}}(x, y^{(1)}) \frac{\partial F_1}{\partial y^{(1)j}}(x, y^{(1)}).$$

We have that $\text{rank}(h_{ij}^{(1)}) = n - 1$ and the variational problem for F_1 uniquely determined the projective class of the geodesic spray S_1 . Within this projective class, S_1 is the only spray whose geodesics are parameterized by the arc-length.

On the second order tangent bundle T^2M , we consider the locally defined functions

$$z^{(2)i}(x, y^{(1)}, y^{(2)}) = y^{(2)i} + \frac{1}{2} \gamma_{jk}^i(x) y^{(1)j} y^{(1)k}.$$

It follows that $z^{(2)i}$ behave as the components of a vector field on M . These components were interpreted as the covariant form of acceleration in [6, (6.5)], as half of the components of the tension field in [9]. It follows that the function $L_2 : T^2M \rightarrow \mathbb{R}$, given by

$$L_2(x, y^{(1)}, y^{(2)}) = \frac{1}{2} g_{ij}(x) z^{(2)i}(x, y^{(1)}, y^{(2)}) z^{(2)j}(x, y^{(1)}, y^{(2)}) = \frac{1}{2} \|z^{(2)}\|_g^2,$$

is a second order regular Lagrangian. The metric tensor of L_2 , given by formula (3.1), is the Riemannian metric g_{ij} . The variational problem for L_2 uniquely determines a semispray $S_2 \in \mathfrak{X}(T^3M)$, whose geodesics are biharmonic curves [9]. We call S_2 the biharmonic semispray and it is uniquely determined by either one of the two equivalent equations (3.14). The local coefficients of the biharmonic semispray can be determined as in [4, (4.6)], while the biharmonic equations can be written as in [4, (4.8)].

For the second order Lagrangian L_2 , the following homogeneity properties are true:

$$(5.4) \quad \mathbb{C}_1(L_2) = 4L_2, \quad \mathbb{C}_2(L_2) = S_1(L_1) = 2g_{ij}(x) z^{(2)i} y^{(1)j}.$$

Using Proposition 3.6 it follows that the biharmonic semispray S_2 satisfies the homogeneity condition $[\mathbb{C}_1, S_2] = S_2$. However, the biharmonic semispray S_2 is not a homogenous semispray since it does not satisfy the equation (2.23) for $\alpha = 3$.

Consider the function $F_2 : T_0^2M \rightarrow \mathbb{R}$,

$$(5.5) \quad F_2 = \frac{4F_1^2 L_2 - (\mathbb{C}_2(L_2))^2}{4F_1^5}.$$

The numerator of the above formula is $\|z^{(2)}\|_g^2 \|y^{(1)}\|_g^2 - g_{ij} y^{(1)i} z^{(2)j} \geq 0$, and hence $F_2 \geq 0$ on T_0^2M . Using the homogeneity properties (5.4) of the Lagrangian L_2 , we obtain that F_2 satisfies the Zermelo conditions (4.1), for $k = 2$. Moreover, the tensor (4.2) that corresponds to F_2 is given by

$$(5.6) \quad h_{ij}^{(2)} = F_2^3 \frac{\partial^2 F_2}{\partial y^{(2)i} \partial y^{(2)j}} = \left(\frac{F_2}{F_1} \right)^3 h_{ij}^{(1)}.$$

It follows that $\text{rank}(h_{ij}^{(2)}) = \text{rank}(h_{ij}^{(1)}) = n - 1$ and therefore F_2 is a Finsler function of order 2. In the Euclidean context, F_2 reduces to the parametric Lagrangian considered by Crampin and

Saunders in [14]. The function $F_2/F_1 \in C^\infty(T_0^2M)$, which relates the angular metrics of the two Finsler functions, is related to the first curvature κ of a curve. Indeed we have $F_2/F_1 = \kappa^2$.

The variational problem for F_2 uniquely determines a system of fourth order differential equations, which is invariant under orientation preserving reparameterizations. By fixing the parameter to be the arc-length, the system reduces to the dynamical equation of motion (38) studied by Matsyuk [22]. In the Euclidean context, a homogeneous semispray, in the projective class determined by the variational problem of F_2 was obtained in [14].

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